

AN APPENDIX TO A PAPER BY B. HANKE AND T. SCHICK

MOSTAFA ESFAHANI ZADEH

ABSTRACT. In this short note we apply methods introduced by B. Hanke and T. Shick to prove the vanishing of (low dimensional-)higher A -genera for spin manifolds admitting a positive scalar curvature metric. Our aim is to provide a short and unified proof for this beautiful result without using the strong Novikov conjecture.

1. INTRODUCTION

Let G be a discrete group and BG its classifying space. The strong Novikov conjecture asserts the injectivity of the rationalized assembly map

$$A: K_*(BG) \otimes \mathbb{Q} \rightarrow K_*(C_{max}^*G) \otimes \mathbb{Q}.$$

Recently B. Hanke and T. Schick have proved the strong Novikov conjecture for those class in $K_*(BG)$ belonging to the dual of $\Lambda^*(G) \subset H^2(BG; \mathbb{Q})$. Here $\Lambda^*(G)$ denote the subring of $H^*(BG; \mathbb{Q})$ generated by elements in $H^2(BG; \mathbb{Q})$ and the duality is defined by means of the homological Chern character $Ch: K_*(BG) \rightarrow H_*(BG; \mathbb{Q})$, c.f. [1, section 11]. Their proof is based on several sophisticated techniques in higher index theory and in particular a fundamental role is played by an assembling techniques that they have invented in [3, 4]. In this short note we are interested in an application of the strong Novikov conjecture. Let M be a closed spin manifold and let $f: M \rightarrow BG$ be any continuous map. It is proved in [6, theorem 3.5] that if the strong Novikov conjecture holds for G and if M admit a metric with everywhere positive scalar curvature then for all $a \in H^*(BG, \mathbb{Q})$,

$$(1.1) \quad \langle \hat{A}(TM) \cup f^*(a), M \rangle = 0.$$

As an immediate corollary of this result, if $f_*([M]) \neq 0$ in $H_*(BG, \mathbb{Q})$ then M does not admit a metric with everywhere positive scalar curvature. For G hyperbolic this corollary was proved by M. Gromov and B. Lawson by means of their relative index theorem (see [2, theorem 13.8]). In this short note we follow [5] to give a proof for vanishing relation (1.1) provided $a \in \Lambda^*(G)$. We do not deal with the strong Novikov conjecture and do not use the assembling construction. This makes all arguments easier to follow and provides a short proof for the vanishing higher genera in low dimension. As the title suggests this note should be considered as an appendix to the paper [5]. The author has merely applied the method of this paper to prove a result which is over looked in [5].

2. THE VANISHING THEOREM

Theorem 2.1. *Let $\Lambda^*(G)$ denote the subring of $H^*(BG; \mathbb{Q})$ generated by elements in $H^2(BG; \mathbb{Q})$. Let M be a closed spin manifold admitting a riemannian metric with positive scalar curvature. Then for all a in $\Lambda^*(G)$*

$$\langle \hat{A}(TM) \cup f^*(a), [M] \rangle = 0.$$

In particular if $f_[M]$ belongs to the dual of $\Lambda^*(G)$ then M cannot carry a metric with positive scalar curvature.*

proof By multiplying with an appropriate integer, we may assume $a \in H^2(M, \mathbb{Z})$. Let $\ell \rightarrow BG$ be the smooth unitary complex line bundle classifying by a . The unitary bundle $L := f^*\ell$ is then classified by $f^*(a)$, i.e. the first Chern class $C_1(L)$ equals $f^*(a)$. Fix a unitary connection on L and denote its curvature by $\omega \in \Omega^2(M)$; a closed differential 2-form which represents $C_1(L)$. Denote by \tilde{M} the covering $\tilde{M} \xrightarrow{\pi} M$ which is classified by f . The discrete group $\Gamma := \pi_1(M)/\ker f_*$ is the Deck

transformation group of \tilde{M} and $\tilde{M}/\Gamma = M$. Let $\tilde{L} := \pi^*(L)$ denote the lifting of L to \tilde{M} . Since the universal cover of BG is contractible, and by naturality, the bundle \tilde{L} is trivial. Fix a unitary isomorphism $\tilde{L} \simeq \tilde{M} \times \mathbb{C}$. With respect to this isomorphism, the lifted connection on \tilde{L} takes the form $d + \eta$ where $\eta \in \Omega^1(\tilde{M})$. Since the structural group of L is abelian, the curvature of this connection is $d\eta$. Though η is not in general Γ -invariant its curvature is and by naturality $d\eta = \pi^*(\omega)$. For $0 \leq t \leq 1$ consider the family $d + t\eta$ of unitary connections on \tilde{L} . The curvature of this family is given by $\tilde{\omega}_t := t\pi^*(\omega)$ and satisfies the following inequality

$$(2.1) \quad \|\tilde{\omega}_t\| \leq c.t .$$

Here c is a constant depending on the geometry of the vector bundle L . Consider the vector bundle

$$\mu := \tilde{M} \times_{\Gamma} \ell^2(\Gamma) .$$

The connection $d + t\eta$ induces a connection ∇_t on this bundle with curvature ω_t . In fact a smooth section of μ may be considered as a L^2 -smooth section of $\tilde{M} \times \mathbb{C}$ and the action of ∇_t on such a section coincides with the action of $d + t\eta$. It is clear from this description that the connection ∇_t is unitary with respect to the inner product of $\ell^2(\Gamma)$ and its curvature ω_t satisfies the following inequality

$$(2.2) \quad \|\omega_t\| \leq c.t ,$$

where c is the constant of (2.1). In [5] the bundle μ and the connection ∇_t are used to construct a Hilbert A_t -module bundle V_t . We describe briefly this construction in below. Fix a point $x \in M$ and a $\tilde{x} \in \tilde{M}$. So one can identify naturally μ_x with $\ell^2(\Gamma)$. Let $y \in M$ be an arbitrary point and γ be a piecewise smooth curve from x to y . Let $A_t(\gamma) \in \text{Hom}(\mu_x, \mu_y)$ denote the parallel translation with respect to ∇_t . For a fix y , denote by $V_t(y)$ the linear-norm completion of the set of all such $A_t(\gamma)$'s. These linear spaces form a smooth vector bundle V_t . Notice that $A_t := V_t(x)$ is in fact a C^* -algebra and has a right module action on each fiber of V_t given by precomposition. Moreover the following pairing

$$(2.3) \quad \langle A_t(\gamma), A_t(\gamma') \rangle := A_t(\gamma)^{-1} \circ A_t(\gamma')$$

defines an A_t -valued bilinear form which supplies V_t with the structure of a Hilbert A_t -module bundle. Let v be an element in μ_y which is the parallel translation of $u \in V_x$ along the curve γ from x to y and let β be a piecewise smooth curve from y to z . Parallel translation of u along $\beta \circ \gamma$ define $v' \in V_{t,z}$. The family $P(\beta)$ given by

$$(2.4) \quad P(\beta) : V_{t,y} \rightarrow V_{t,z} ; \quad P(\beta)(A(\gamma))(v) = v'$$

satisfy the properties of parallel translation. Clearly these parallel translations are A_t -linear and are unitary with respect to (2.3). Therefore they define an A -linear unitary connection on V_t whose curvature Ω_t is given by

$$(2.5) \quad \Omega_t(y)(A) = A \circ \omega_t(y) , \quad A \in V_{t,y}$$

and satisfies again the following inequality

$$(2.6) \quad \|\Omega_t\| \leq c.t ,$$

The algebra A_t has a complex-valued natural trace τ_t given by the following formula, c.f. [5, Lemma 2.2]

$$\tau_t(A_t(\gamma)) = \langle A_t(\gamma)(1_e), 1_e \rangle .$$

Notice that the identification $\mu_x = \ell^2(\Gamma)$ is used here. Let D^{V_t} denote the spin Dirac operator on M twisted by the bundle V_t . The Mishchenko-Fomenko index $\text{ind } D^{V_t}$ takes its value in $K_0(A_t)$ which is acted on by τ_t . The following index formula gives a topological expression for $\tau_t(\text{ind } D^{V_t})$ (see [7, Theorem 6.9])

$$\tau_t(\text{ind } D^{V_t}) = \langle \hat{A}(TM) \cup \text{Ch}_{\tau_t}(V_t), [M] \rangle ,$$

where $\text{Ch}_{\tau_t}(V_t) = \tau_t(\exp(\Omega_t))$. From (2.5) one has $\tau_t(\Omega_t) = \tau_t(t\pi^*f^*(a)) = tf^*(a)$, so

$$\text{Ch}_{\tau_t}(V_t) = 1 + tf^*(a) + t^2 f^*(a) \wedge f^*(a) + \dots$$

is a polynomial function of t of degree at most $m = \dim M$ with coefficient in $\Lambda^* T^* M$. Consequently

$$\tau_t(\text{ind } D^{V_t}) = \langle \hat{A}(TM), [M] \rangle + t \langle \hat{A}(TM) \cup f^*(a), [M] \rangle + \cdots \in \mathbb{R}[t].$$

Inequality (2.6) and the Lichnerowicz formula imply the vanishing of $\text{ind } D^{V_t}$ for small values of t . The vanishing of above polynomial for small values of t implies the vanishing of all its coefficients. In particular

$$\langle \hat{A}(TM) \cup f^*(a), [M] \rangle = 0,$$

which is the desired relation and completes the proof of the theorem for $a \in H^2(M, \mathbb{Z})$. Now let $a = a_1 \cup \cdots \cup a_k$ where each a_j is an element in $H^2(BG, \mathbb{Q})$. Consider the line bundle $\mathcal{L} := L_1 \otimes \cdots \otimes L_k$ where L_j is the line bundle classified by $f^*(a_j)$. Proceeding as in above by using the line bundle \mathcal{L} (instead of L) and the connection $d + t_1 \eta_1 + \cdots + t_k \eta_k$ (instead of $d + t\eta$), one deduces the vanishing of a polynomial function of variables t_j for $1 \leq j \leq k$. The coefficient of $t_1 t_2 \cdots t_k$ in this polynomial is

$$\langle \hat{A}(TM) f^*(a_1 \cup \cdots \cup a_k), [M] \rangle.$$

Therefore the above expression must be zero. This complete the proof of the theorem. \square

REFERENCES

- [1] Paul Baum and Ronald G. Douglas. K homology and index theory. In *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, volume 38 of *Proc. Sympos. Pure Math.*, pages 117–173. Amer. Math. Soc., Providence, R.I., 1982.
- [2] Mikhael Gromov and H. Blaine Lawson, Jr. Positive scalar curvature and the Dirac operator on complete Riemannian manifolds. *Inst. Hautes Études Sci. Publ. Math.*, (58):83–196 (1984), 1983.
- [3] B. Hanke and T. Schick. Enlargeability and index theory. *J. Differential Geom.*, 74(2):293–320, 2006.
- [4] Bernhard Hanke and Thomas Schick. Enlargeability and index theory: infinite covers. *K-Theory*, 38(1):23–33, 2007.
- [5] Bernhard Hanke and Thomas Schick. The strong Novikov conjecture for low degree cohomology. *Geom. Dedicata*, 135:119–127, 2008.
- [6] Jonathan Rosenberg. C^* -algebras, positive scalar curvature, and the Novikov conjecture. *Inst. Hautes Études Sci. Publ. Math.*, (58):197–212 (1984), 1983.
- [7] Thomas Schick. L^2 -index theorems, KK -theory, and connections. *New York J. Math.*, 11:387–443 (electronic), 2005.

MOSTAFA ESFAHANI ZADEH.

MATHEMATISCHES INSTITUT, GEORG-AUGUST-UNIVERSITÄT,
GÖTTINGEN, GERMANY AND
INSTITUTE FOR ADVANCED STUDIES IN BASIC SCIENCES,
ZANJAN, IRAN

E-mail address: zadeh@uni-math.gwdg.de